Designing Algorithms with Divide-and-Conquer

Lecture 06.03

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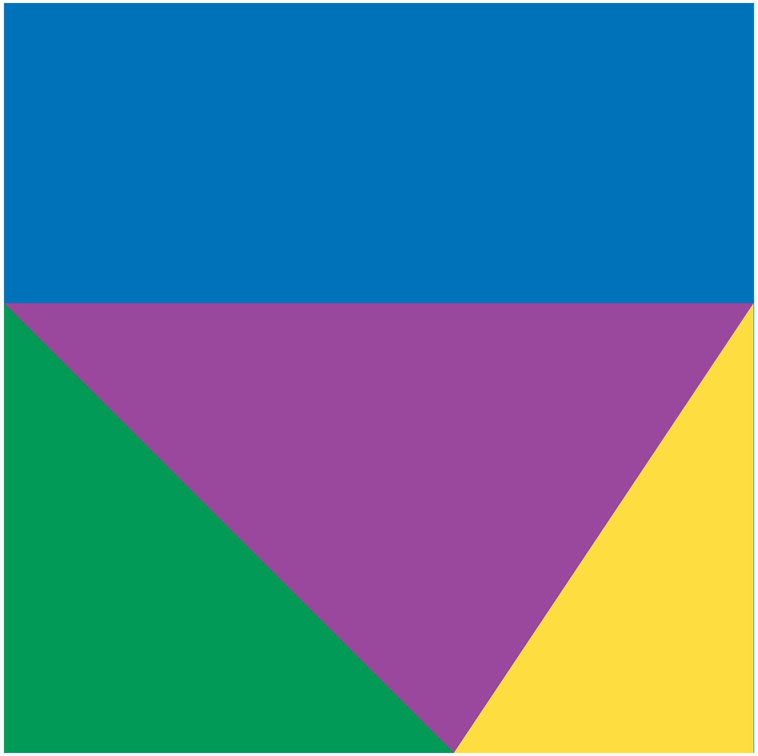
Main algorithm design strategies

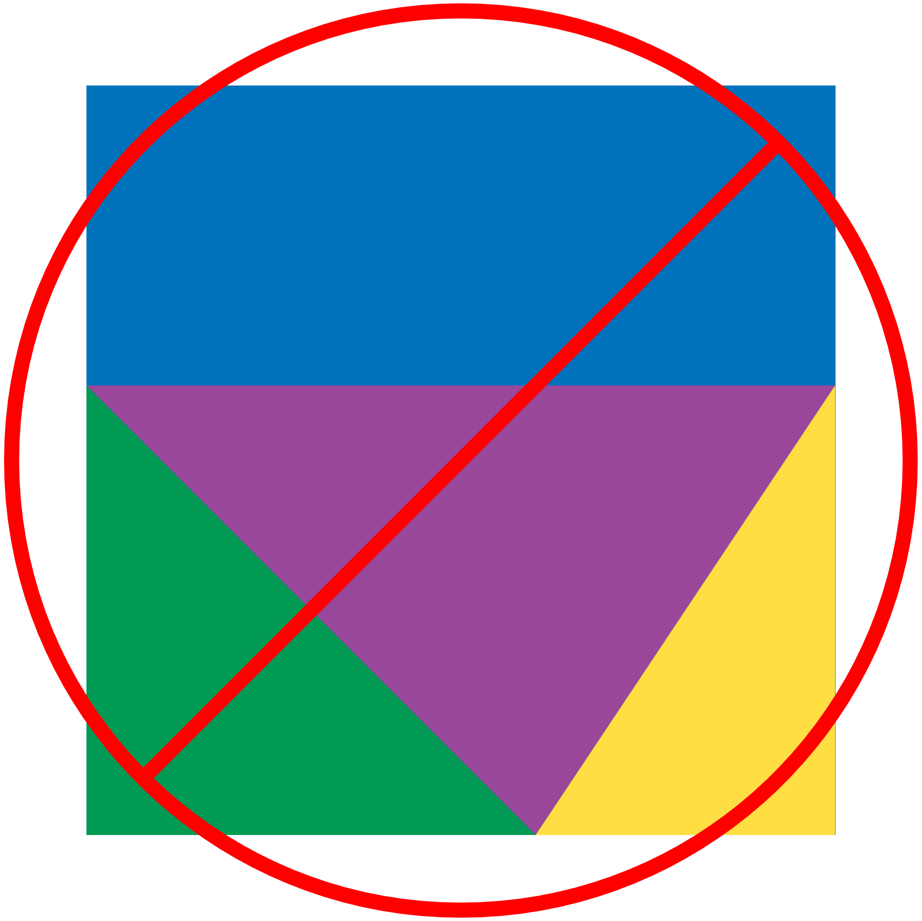
- ✓ ***Exhaustive Computation*.** Generate every possible candidate solution and select an optimal solution.
- ✓ ***Greedy. Create next candidate solution one step at a time by using some greedy choice.***
- ● ***Divide and Conquer*.** Divide the problem into non-overlapping subproblems of the same type, solve each subproblem with the same algorithm, and combine sub-solutions into a solution to the entire problem.
- ● ***Dynamic Programming*.** Start with the smallest subproblem and combine optimal solutions to smaller subproblems into optimal solution for larger subproblems, until the optimal solution for the entire problem is constructed.

Big problem to be solved

Divide: Break into <u>non-overlapping</u> subproblems of <u>the same type</u>

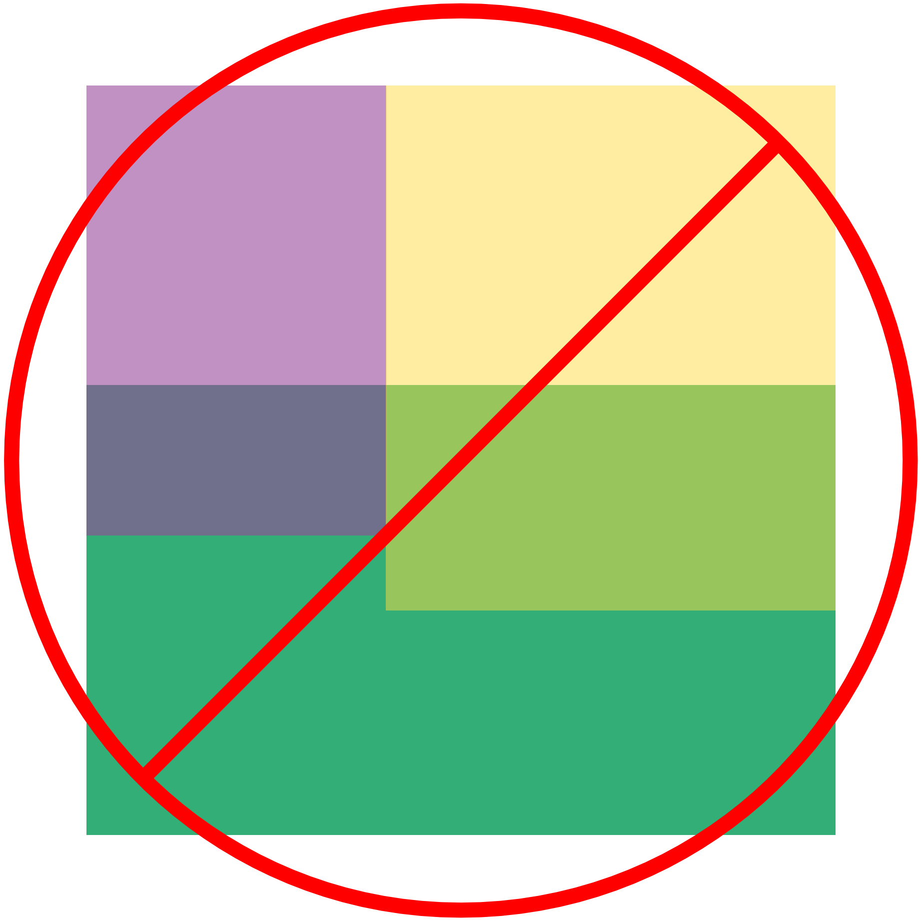
Problem





not the
same type

Problem



overlapping

Divide-and-conquer steps

1. Break into *non-overlapping* subproblems *of the same type*
2. Solve subproblems
3. **Combine results**

Most difficult!

Two examples:

- Counting inversions
- Closest pair

Counting inversions

Motivation

- Music site tries to match user song preferences with others.
- I rank *n* songs.
- Music site consults database to find people with similar tastes.

songs

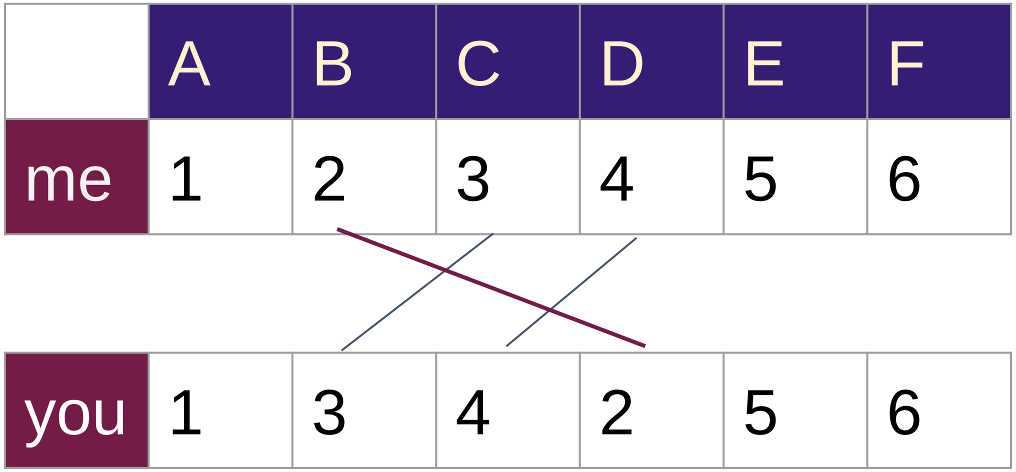
	A	B	C	D	E	F
me	1	2	3	4	5	6
you	1	3	4	2	5	6

How similar are me and you?

SImilarity of rankings

- Similarity metric:
  number of *inversions* between two rankings.
- My rank: 1,2,3,4,5,6
- Your rank: 1,3,4,2,5,6
  - for the same songs

songs

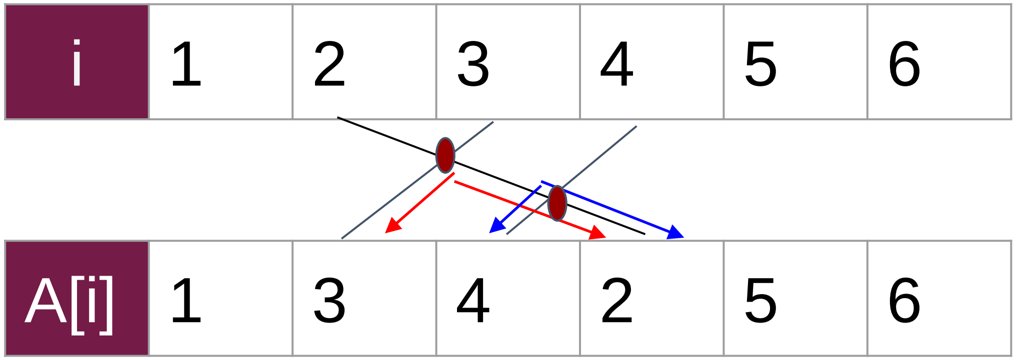


	A	B	C	D	E	F
me	1	2	3	4	5	6
you	1	3	4	2	5	6

For a perfect match you should have ranked D at 4, but you ranked it at 2

Definition

An *inversion* is a pair (A[i], A[j]) of array elements such that index i<j and A[i] > A[j]



i	1	2	3	4	5	6
A[i]	1	3	4	2	5	6

2 inversions in total:
(3,2) and (4,2)

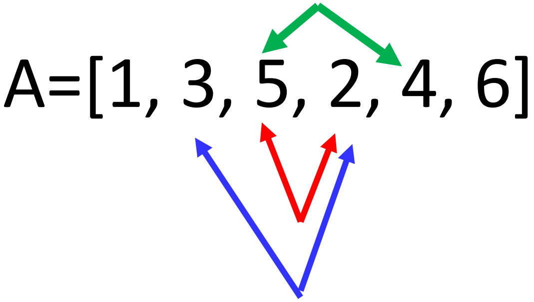
Problem: counting inversions

Input: an array *A* of length *n* with numbers 1,2,…*n* in some order

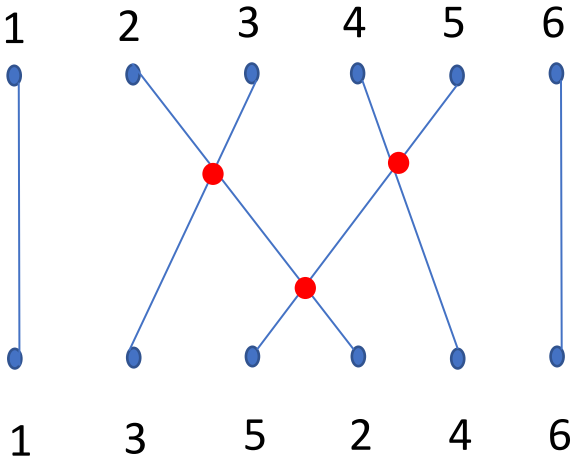
Output: number of *inversions*: number of pairs *A*[*i*],*A*[*j*] of array elements with *i*<*j* and *A*[*i*] > *A*[*j*]

- If A is sorted – what is the number of inversions?
- What is the number of inversions if A is reversed?
- What is the number of inversions in A=[1,3,5,2,4,6]?

Example



- A=[1, 3, 5, 2, 4, 6]



- Inversions:

(3,2), (5,2), (5,4)

What is the largest-possible number of inversions that a 6-element array can have?

Brute-force algorithm for counting inversions

Algorithm count_naive (array A of n integers)

```
count:= 0
for i from 1 to n-1:
        for j from i+1 to n:
                if A[j] < A[i]
                        count:= count + 1
return count
```

Complexity?

Can we do better?

But how can we do better if total number of inversions is $O(n^2)$???

Idea 1: Divide + Conquer

After dividing array into 2 halves, n/2 each:

For each (i,j) recursively determine if (A[i],A[j]) is an inversion

There are 3 possible cases (3 types of inversions):

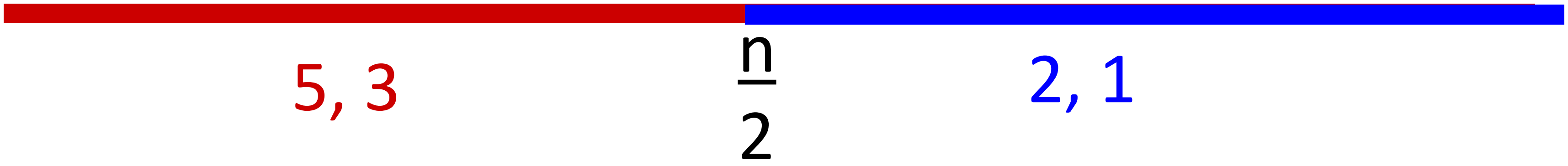
Left inversions : if i,j <= n/2

Right inversions: if i,j > n/2

These two can be computed recursively

Split inversions : if i<=n/2 and j > n/2

But how to compute these?



Developing recursive algorithm

count (array A of length n)

```
if n=1
        return 0
Else
        x = count (1st half of A, n/2)
        y = count (2nd half of A, n/2)
        z = count_split_inv(A, n)
return  x+y+z
```

We do not know how to do that

If we manage to do *CountSplitInv* in O(n) time

then *Count* will run in O(n log n) - just like Merge Sort

Idea 2. What if we use *merge* from merge sort?

- ❑ Have recursive calls both *count inversions and sort*

- ❑ It turns out that the *merge* subroutine **automatically** recovers inversions!

Recursive Algorithm (in progress)

sort_count (array A of length n)

```
if n=1
    return (A,0)
Else
    (B, x) = sort_count (1st half of A, n/2)
    (C, y) = sort_count (2nd half of A, n/2)
    (D, z) = count_split_inv(B,C)
return (D, x+y+z)
```

B- sorted 1st half of A

C- sorted 2nd half of A

We still do not know how to do that

If we manage to do *count_split_inv* in O(n) time then *sort_count* will run in O(*n log n*) - just like Merge Sort

merge subroutine: from Merge Sort

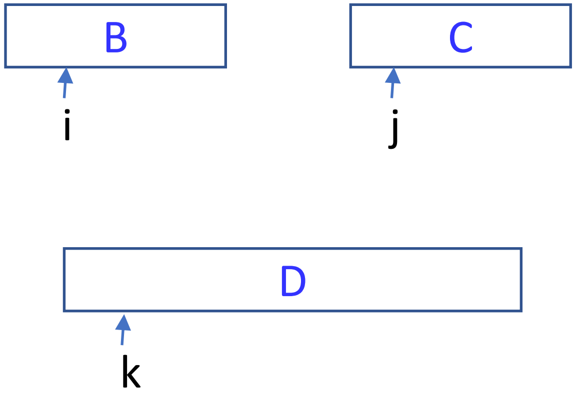
D = will contain sorted array

B = 1st sorted subarray [1:n/2]

C = 2nd sorted subarray [n/2:n]

i = 1

j = 1

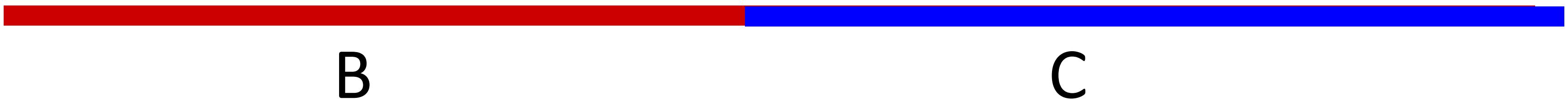


```
for k: = 1 to n
        if B[i]< C[j]
                D[k]: = B[i]
                i:= i+1
        else if C[j] < B[i]
                D[k]: = C[j]
                j:= j+1

…
```

Stop and think

Suppose the input array A **has no split inversions**.



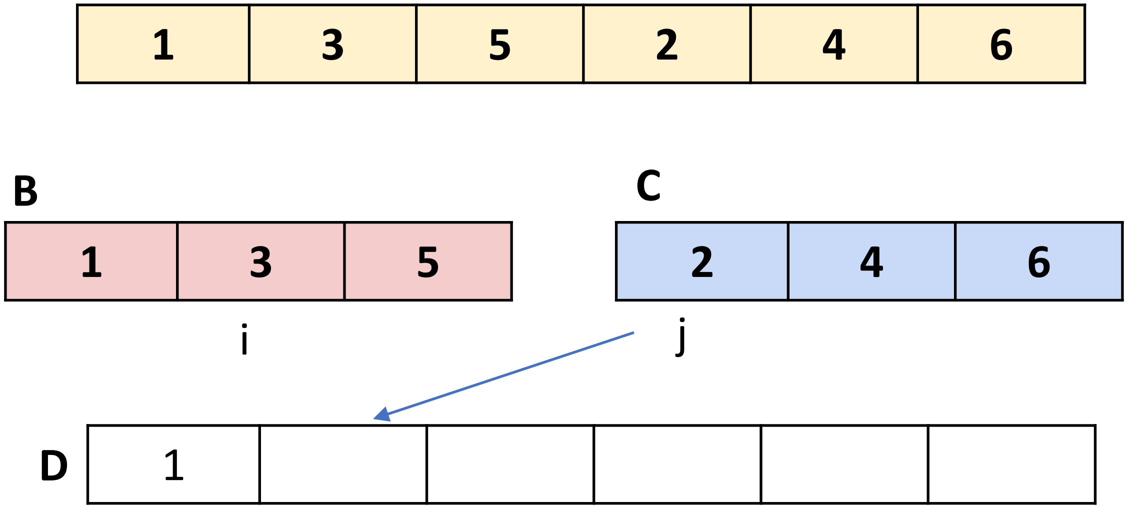
What is the relationship between the sorted subarrays B and C?

A. B has the smallest element of A, C has the second--smallest, B has the third- smallest, and so on.

A. All elements of B are less than all elements of C.

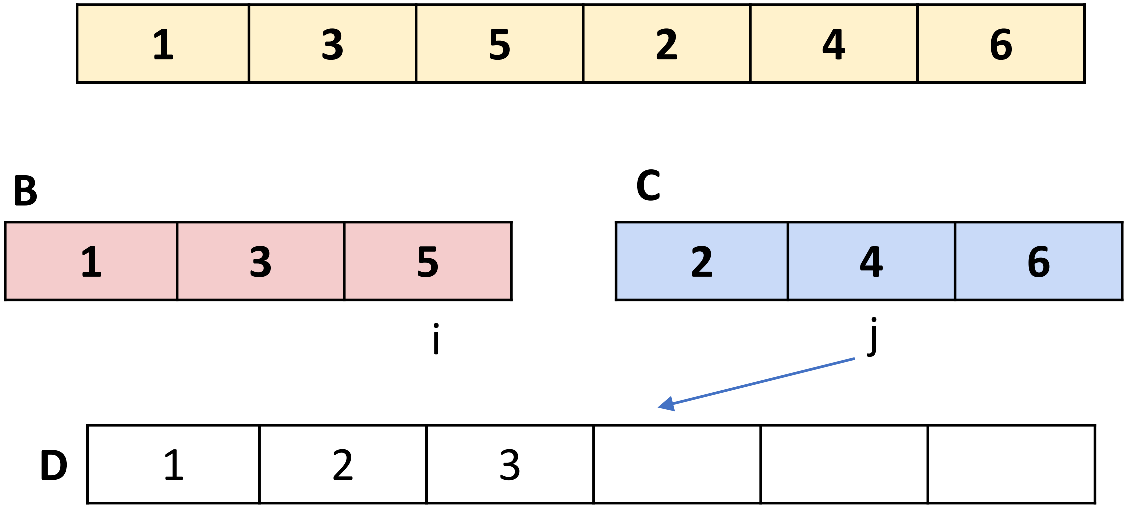
A. There is not enough information to answer this question.

Sample merge



Discovered 2 inversions:
(3,2) and (5,2)

Sample merge



Discovered inversion
(5,4)

General claim

The split inversions involving an element *y* of the 2nd array *C* are precisely the numbers left in the 1st array B when *y* is copied to the output *D*.

Proof:

Let *x* be an element of the 1st array B.

- If x copied to output D before y, then x < y

  => no inversions involving x and y

- If y copied to output D before x, then y < x

  ⇒ x and all elements after it are (split) inversions. ■

Recursive Algorithm (revised)

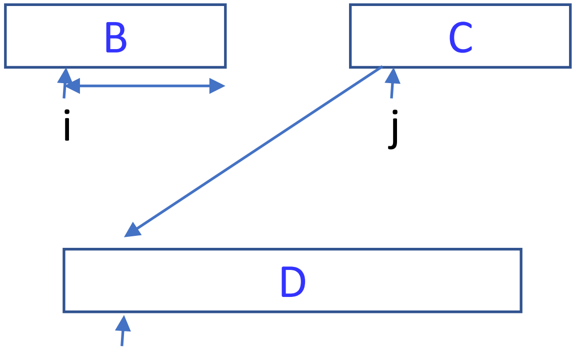
sort_count_inv (array A of length n)

```
if n=1
        return (A, 0)
Else
        (B, x) = sort_count_inv(1st half of A)
        (C, y) = sort_count_inv(2nd half of A)
        (D, z) = merge_count_split_inv(B,C)
return  (D, x+y+z)
```

Split inversions are recovered during the merge of the sorted sub-arrays

Merge and count

- While merging the two sorted subarrays, keep running total of number of split inversions



- When element of 2nd array *C* gets copied to output *D*, increment *total* by number of elements remaining in 1st array *B*

merge running total

Runtime of *merge_count_split_inv*: $\underbrace{O(n)}_{\text{merge}} + \underbrace{O(n)}_{\text{running total}} = O(n)$

***sort_count_inv* runs in O(n log n) time**

just like Merge Sort

Closest pair

Motivation

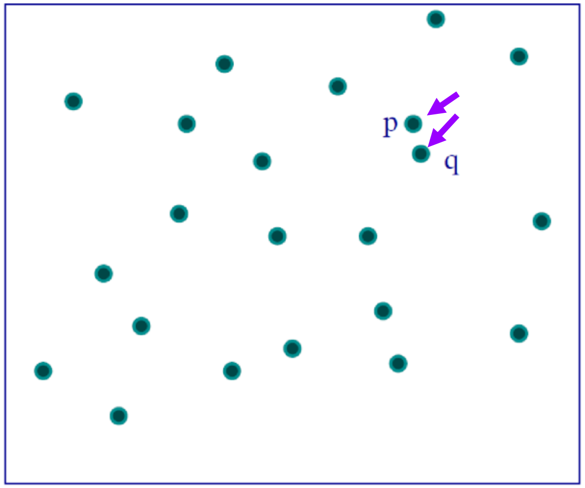
The closest-pair is a subroutine for:

- Dynamic minimum spanning trees
- Straight skeletons and roof design
- Ray-intersection diagram
- Collision detection applications
- Hierarchical clustering
- Traveling salesman heuristics
- Greedy matching
- …

> *"A pair of the closest points, the one lying on a robot and the other on its obstacles, yields the most important information for generation of obstacle-avoiding robot motions." ref*

Closest Pair Problem

- **Input**: *n* points in *d* dimensions
- **Output**: two points *p* and *q* whose mutual distance is smallest



A naive algorithm takes $O(dn^2)$ time.

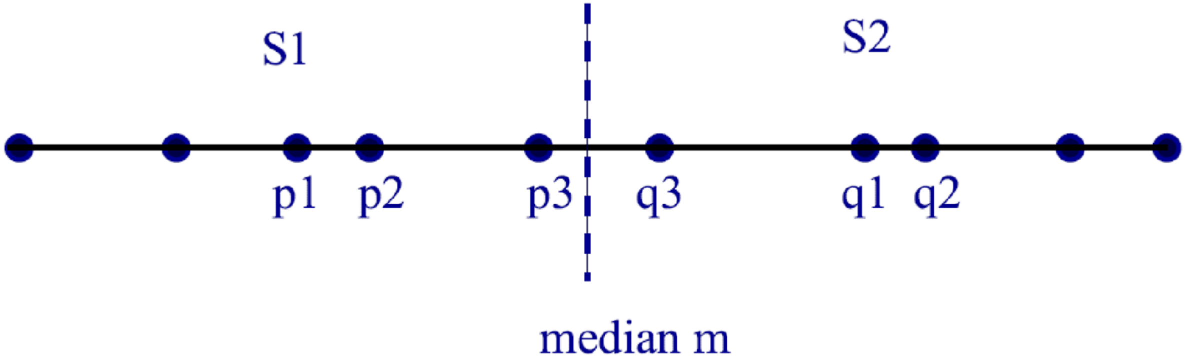
(Number of dimensions *d* can be assumed a constant for a given problem)

Can we do better?

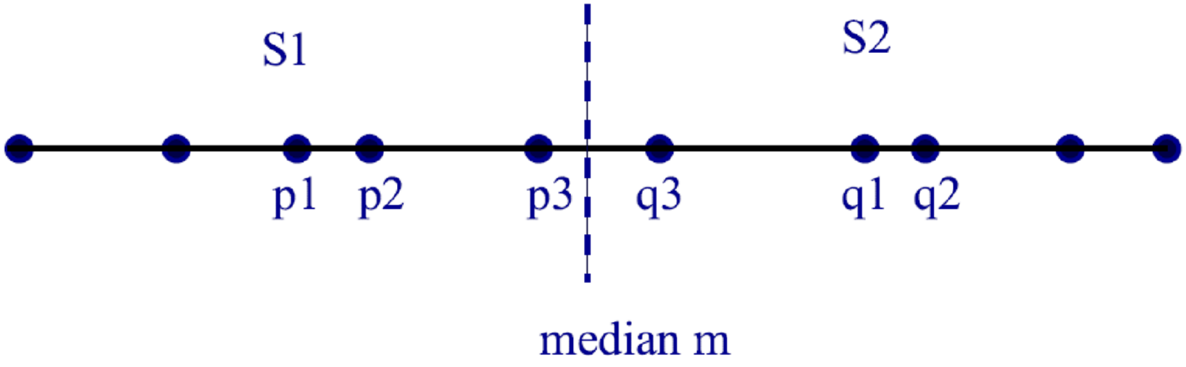
Closest pair in one dimension

Can be solved in O(*n* log*n*) via sorting, and then linear scanning.
Let's develop a **recursive** solution to find the closest pair

- If the points are sorted by their coordinate:
- Divide the points set S into 2 sets S_1, S_2, by median x-coordinate m such that $p<q$ for all $p \in S_1$ and $q \in S_2$
- Recursively compute closest pair (p_1,p_2) in S_1 and (q_1,q_2) in S_2



Closest pair in one dimension: *combine* step



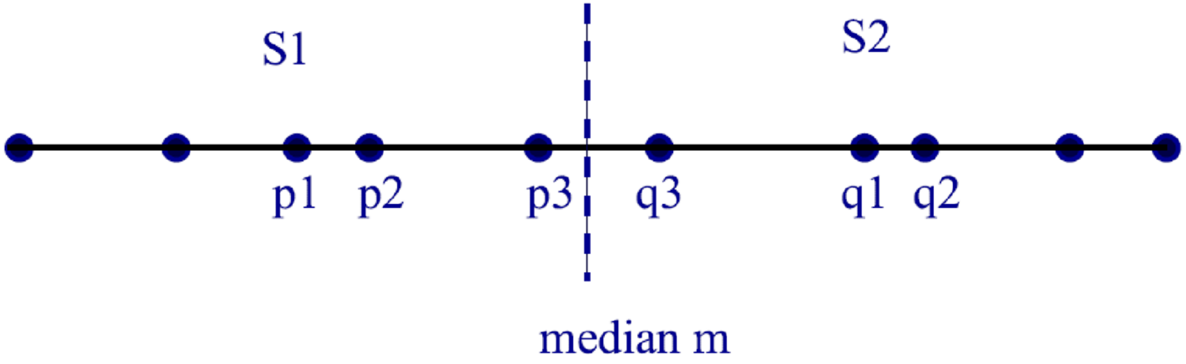
- Let δ be the smallest pairwise distance found in 2 partitions

δ = min($|p_2 – p_1|$, $|q_2 – q_1|$

- The closest pair is either (p_1,p_2), or (q_1,q_2), or some (p_3,q_3) where $p_3 \in S_1$ and $q_3 \in S_2$

- Can we find (p_3,q_3) in a constant time?

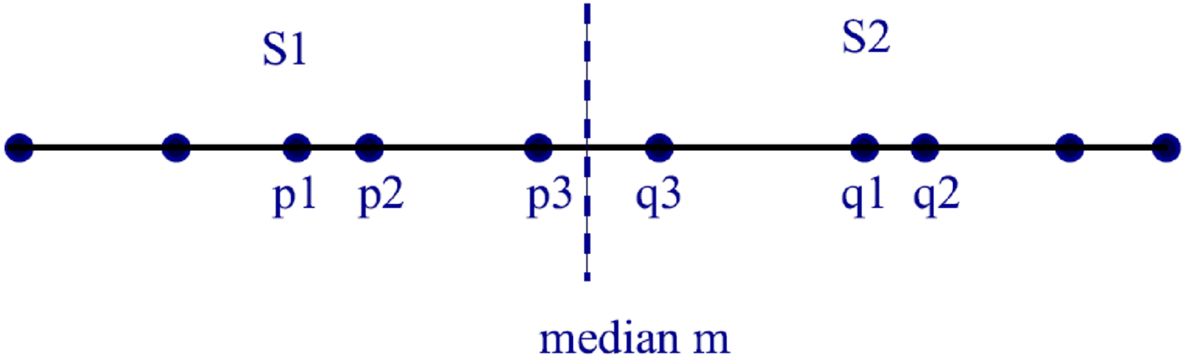
Closest pair in 1 dimension



- The closest pair is either (p_1,p_2), or (q_1,q_2), or some (p_3,q_3) where $p_3 \in S_1$ and $q_3 \in S_2$

- Key observation: If *m* is the dividing coordinate, then both p_3 and q_3 have to be within δ of *m*

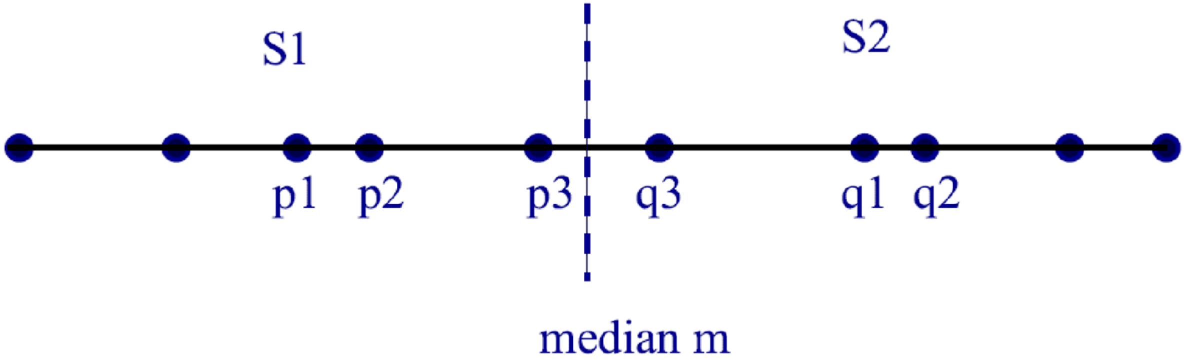
Closest pair in 1 dimension



- Key observation: If *m* is the dividing coordinate, then both p_3 and q_3 have to be within δ of *m*

- How many such pairs exist?

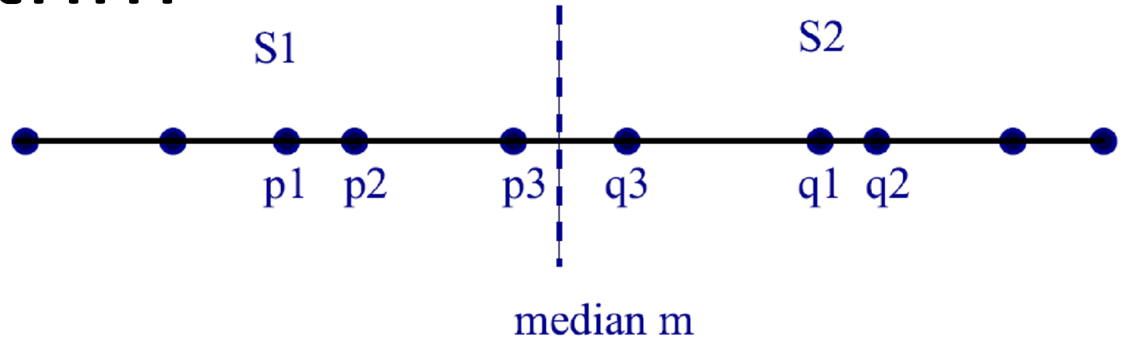
Closest pair in 1 dimension



- Key observation: If *m* is the dividing coordinate, then both p_3 and q_3 have to be within δ of *m*

- How many points of S1 can lie in the interval (m – δ, m]?
- So we need to check one pair only - constant time

Closest pair 1D: recursive algorithm



```
closest_pair (S – set of sorted points $p_i…p_n$, $n>=2$)
    if $|S| = 2$
        return $\delta = |p_2 - p_1|$

    Divide $S$ into $S_1$ and $S_2$ at m = value[n/2]
    $\delta_1$ = closest_pair ($S_1$)
    $\delta_2$ = closest_pair ($S_2$)
    $\delta_3$ = closest_pair_across ($S_1$, $S_2$, min($\delta_1$, $\delta_2$))
    return $\delta$ = min($\delta_1$, $\delta_2$, $\delta_3$)
```

Here we only compute the shortest distance, but it is easy to modify to return 2 points which produced this distance

Constant time

Closest pair in 1 dimension: time complexity

closest_pair (S – set of sorted points $p_i \ldots p_n$, $n>=2$)

```
if $|S| = 2$
    return $\delta = |p_2 - p_1|$
Divide $S$ into $S_1$ and $S_2$ at m = value[n/2]
$\delta_1$ = closest_pair ($S_1$)
$\delta_2$ = closest_pair ($S_2$)
$\delta_3$ = closest_pair_across ($S_1$, $S_2$, min($\delta_1$, $\delta_2$))
return $\delta$ = min($\delta_1$, $\delta_2$, $\delta_3$)
```

Constant time

$T(n) = 2T(n/2) + O(1)$

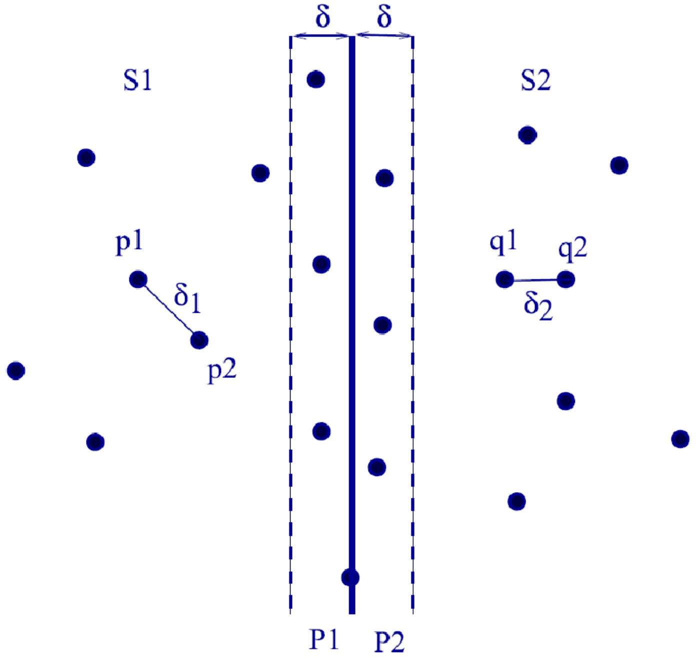
Which solves into $O(n)$

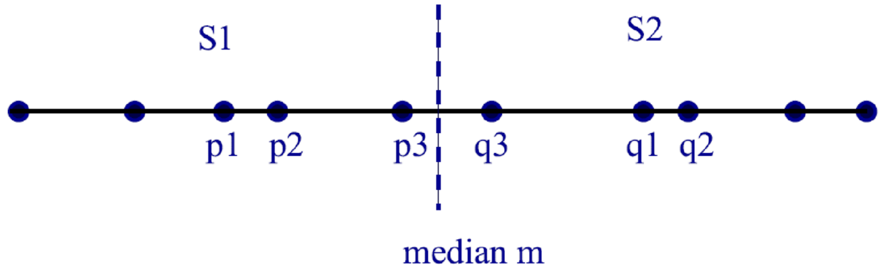
We will learn why later

Together with sorting: $O(n \log n)$

Closest pair in 2 dimensions

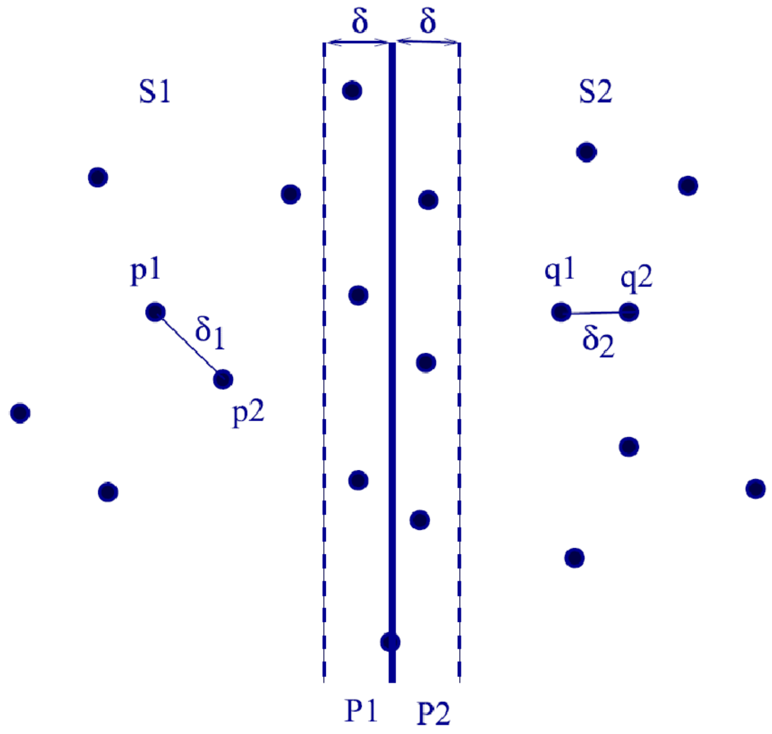
The previous algorithm does not generalize to higher dimensions, **or does it**?





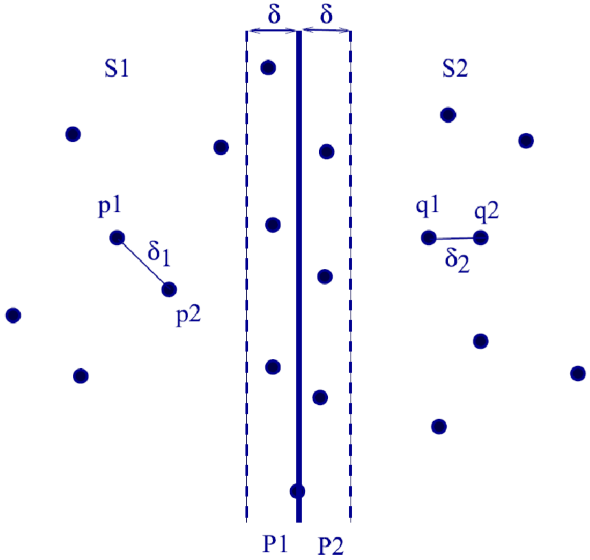
2D closest pair: divide

- Taking sorting as a free O(n log n) invariant, we sort all points in S by x coordinate
- Partition S into S_1, S_2 by vertical line l defined by median x-coordinate in S



2D closest pair: conquer

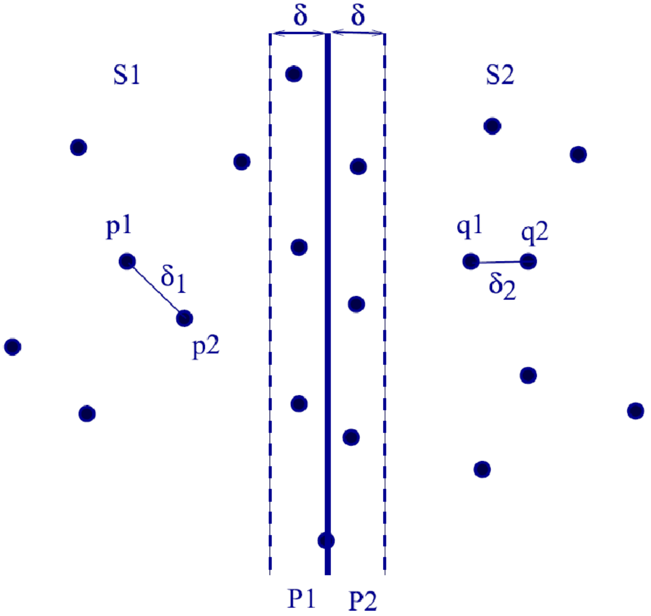
- Recursively compute closest pair distances δ_1 and δ_2 in S_1 and S_2
- Set $\delta=\min(\delta_1, \delta_2)$



2D closest pair: combine

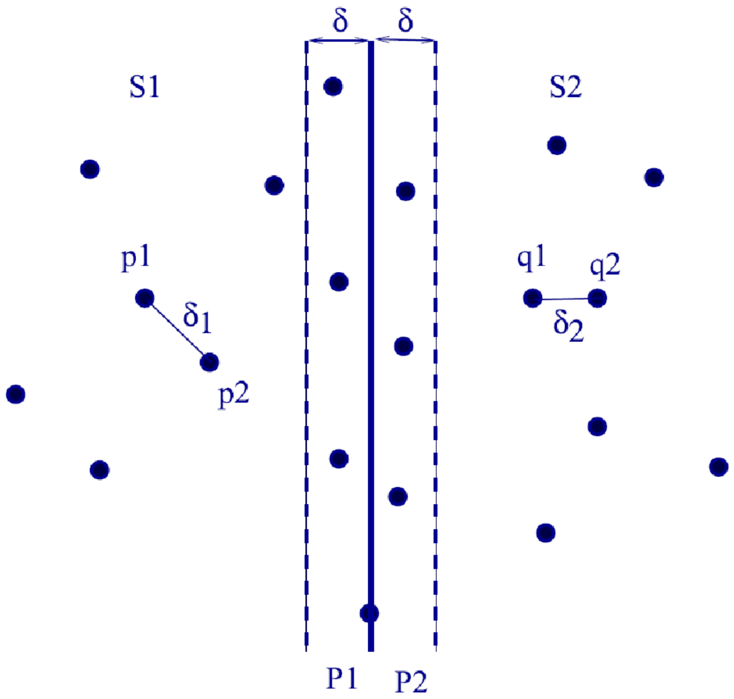
- Closest pair distances in S_1 and S_2 are δ_1 and δ_2.
  $\delta = \min(\delta_1, \delta_2)$
- Now need to combine: compute the closest pair across dividing line *l*
- In each candidate pair (p,q), where $p \in S_1$ and $q \in S_2$,

the only candidate points *p*, *q* must both lie within δ of *l*.



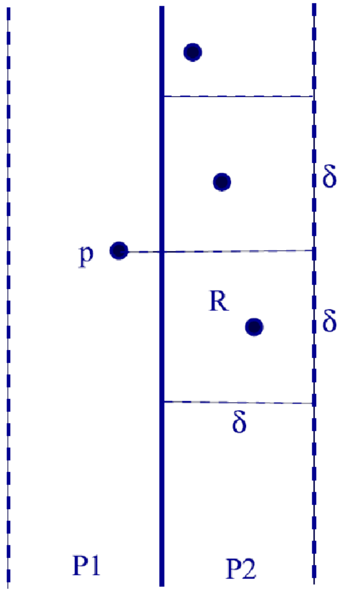
2D closest pair combine: complications

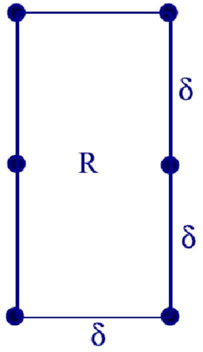
- At this point, complications arise, which were not present in 1D
- It is entirely possible that all *n*/2 points of S_1 (and S_2) lie within δ of *l*
- Naïvely, this would require $n^2/4$ comparisons



Combining split points

- Consider a point *p* ∈ S_1.
- All points of S_2 within distance δ of *p* must lie in a δx2δ rectangle *R*
- How many points can be inside *R* if we know that each pair is at least δ apart?
- **In 2D, this number is at most 6!**





So we only need to perform (*n*/2)*6 distance calculations during the combine step!
We do not have the O(*n* log *n*) algorithm yet. Why?

Combine in linear time

- In order to determine at most 6 potential mates of *p*, project *p* and all points of S_2 into *y* axis
- Pick out points whose projection is within δ of *p*: at most 6

- If we pre-sort S_1 and S_2 by the *y* coordinate
- Then we can do our check for all p $\in$ S_1, by walking sorted lists S_{1y} and S_{2y}, in total O(*n*) time

The entire solution then runs in **O(*n* log *n*)**

https://www.geeksforgeeks.org/closest-pair-of-points-using-divide-and-conquer-algorithm/